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ABSTRACT

This paper has two goals. The first is to present a model of the acquisition of a concept image of function. Theories describing the objectification of function are outlined through two different but related paths, and both stem from the conception of function as a process. The first path to objectification involves the generalization of the process view into one based on relationships between sets. The second path deals with function as an object possessing various properties, such as extrema and continuity. The second goal is to present the results of a year-long investigation on the acquisition of a function concept image in a high school Algebra II course which made significant use of the graphing calculator (GC). It was found that the students were able to define function objectively, but discussions and problem solving behaviors suggesting objectification occurred to a lesser degree. The graphic representation allowed objectification to occur more readily, and the use of the GC appeared to facilitate students' translation ability between functional representations. Students increased their use of the GC throughout the year, particularly on tasks which required graphical thinking. Misconceptions directly resulting from the use of the GC were also identified. (Contains 33 references.) (Author)

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The Effect of Graphing Calculators on Students' Conceptions of Function

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Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April, 1994

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ABSTRACT

This paper has two goals. The first is to present a model of the acquisition of a concept image of function. Theories describing the objectification of function are outlined through two different but related paths, and both stem from the conception of function as a process. The first path to objectification involves the generalization of the process view into one based on relationships between sets. The second path deals with function as an object possessing various properties, such as extrema and continuity.

The second goal is to present the results of a year-long investigation on the acquisition of a function concept image in a high school Algebra II course which made significant use of the graphing calculator (GC). It was found that the students were able to define function objectively, but discussions and problem solving behaviors suggesting objectification occurred to a lesser degree. The graphic representation allowed objectification to occur more readily, and the use of the GC appeared to facilitate students' translation ability between functional representations. Students increased their use of the GC throughout the year, particularly on tasks which required graphical thinking. Misconceptions directly resulting from the use of the GC were also identified.

INTRODUCTION

Recent discussions of algebraic instruction by educational researchers have centered on reconsidering the central topics and features of instruction at this level (NCTM, 1989; Brown et. al., 1988). Studies have shown that traditional instruction is lacking in the development of algebraic concepts (Brown et. al., 1988; Kieran, 1992). The secondary school results from the fourth NAEP mathematics assessment (Brown et. al., 1988) state that students view mathematics in

a procedural, rule-based manner. Further, they state that:

> Secondary school students are not able to apply (algebraic) knowledge in problem-solving situations, nor do they appear to understand many of the structures underlying these mathematical concepts and skills.

The Curriculum Standards (NCTM, 1989) of the National Council of Teachers of Mathematics echo these remarks. However, despite these efforts, the content and instructional techniques of high school algebra have changed very little over the years (Kieran, 1992).

The increased availability of mathematical instructional software, computer games, and graphing packages has prompted the use of more visual instructional approaches to algebra. Because educators are now realizing the differences between graphing utilities and educational software (Goldenberg, 1991), these technological aids provide the mathematics community with a chance to incorporate visually-based problem solving tools into the classroom. *The Algebra Toolkit* (Schwartz, Yerushalmy, and Harvey, 1991), *Function Probe* (Confrey, 1991), and numerous other computer graphing utilities are being devised, implemented, and monitored by researchers as part of a new wave of more modern mathematics instruction. We are presently at the dawn of this movement, and information is vitally needed to help in our understanding of the benefits and weaknesses of this form of instruction.

Kaput (1989) maintains that visual representations of computer technologies can be intertwined with other representations to allow for innovative, effective teaching. Evidence exists that these techniques can be effective in developing structural understandings of algebra (Romberg, Carpenter, and Fennema, 1993). Limitations of graphing technologies exist, as the visual representations can lead to misconceptions, particularly the common "graph as picture" interpretation (Goldenberg, 1988; Mokros and Tinker, 1987). Evidence also exists (Slavit, 1993) that students who use graphing technology can treat the graph as an additional representation rather than as a more central part of their function concept image. However, the students in this study had prior training through more traditional means, and evidence was provided to indicate that this was a factor in these conceptions.

One popular instructional device is the portable and relatively inexpensive graphing calculator. The graphing calculator (GC) facilitates the production of a graphic representation, and it allows for numeric problem solving techniques. Key potential benefits of the GC include its portability and the ability to present and analyze a given function in a multi-representational framework. The relative inexpensiveness of several models of graphing

calculators, along with their portability, has made the GC quite popular in high school and college classrooms.

The graphing calculator has been available to the mathematics community for approximately ten years. There exists only a small amount of research regarding its effectiveness on learning and instruction (Leinhardt et al, 1990), but available evidence suggests that the GC has a significant impact on the development of concepts (Estes, 1990; Rich, 1990; Teles, 1989). Ruthven (1990) found some evidence that the GC promotes the recognition of classes of functions both graphically and symbolically, but this may occur only with extended use. Apparently, students introduced to graphing calculators for only a short period of time can have difficulty adjusting to the instruction (Giamati, 1990).

One of the potential benefits of using the GC in algebra instruction is that its visual capabilities may support the development of important functional concepts. If we think of understanding as building networks between related concepts (Davis, 1984, 1992; Hiebert and Carpenter, 1992), an understanding of function would involve translating information across different representational systems. To understand how this development may occur, it is helpful to consider the ways in which function can be conceived.

Process View of Function

Evidence exists that students initially acquire a *process view* of function (Sfard, 1989, 1991). This involves function as a linear operation which exists dynamically. Procedural conceptions of function involve actions such as an arithmetic process, a "function machine", or an algorithmic view.

Structural View of Function

The process view can evolve into a more permanent *structural view*. The following description of the structural view involves 1) relationships between sets, and 2) an awareness of specific functional properties which exist in a variety of representations. These two views

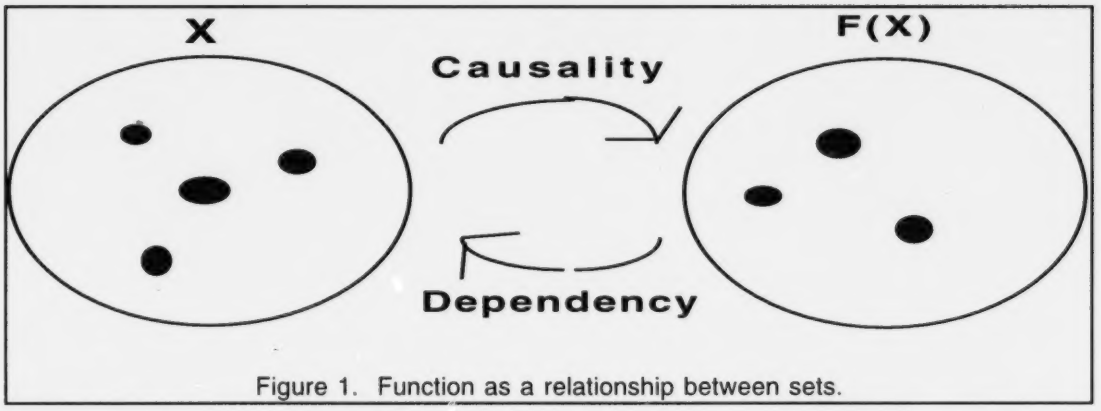


Figure 1. Function as a relationship between sets.

describe routes through which students may objectify the function concept, and we may think of these different views as various pieces of the student's overall concept image (Vinner, 1983). What is essential in a structural view is that function is thought of as an object which can be manipulated and acted upon (Dubinsky and Harel, 1992; Kaput, 1992; Moschkovich, Schoenfeld, and Arcavi,1993; Sfard, 1991, 1992).

View 1: Function as a Dependency Relation Between Two Sets. In a *relational view*, function is thought of in terms of the correspondences between sets of inputs and outputs. No longer is function thought of solely in terms of the production of individual outputs. Instead, the dependence relationship between domain and range "connects" these sets into a single construct. This single construct can then in turn be operated on (such as through a translation).

A relational view involves the notions of causality and dependency. There is evidence that children as young as six years old possess some qualitative notions of function based on causality (Piaget et. al., 1977; Case, 1982; Halford, 1982). For example, a child can understand how a weight causes a string to stretch. These "left-to-right" conceptions (x turns into y) are appropriate at this intuitive level, but "right-to-left" conceptions (y depends on x) are also needed to understand the relational view of function (see Figure 1).

View 2: Function as an Object Possessing Various Properties. Through an intensive look at elementary functions, students may understand function to be a related set of procedures and functional properties in a variety of functional representations. These representations are commonly a symbolic equation, graph, and table of values. The procedures performed on functions give rise to an understanding of functional properties. These properties can be specific to a function class (such as linear slope) or generalize to several function classes (such as symmetry). A student can then conceive of function as an object possessing or not possessing these properties. I hypothesize that this *property-oriented view* develops in three stages which correspond to the instructional sequencing of a typical elementary functions course.

Stage 1: Equivalence of Procedures Across Representations. The first stage of the functional property view involves an ability to realize the equivalence of procedures which exist in different representations. Prerequisite to this ability are translation skills (Janvier, 1987). Noting that the processes of symbolically solving $f(x) = 0$ and graphically finding x-intercepts are structurally equivalent demonstrates this awareness. It is here that the student first begins to deeply understand the underlying mathematical purposes of the procedures by making connections between analogous procedures across different representations.

Stage 2: Equivalence of procedures across function classes. A view of function as a related set of properties also involves the ability to generalize procedures across classes of functions. Students at this second stage can translate procedures across representations (Stage 1), but are also beginning to realize that some of these procedures have analogues in other function classes. For example, the symbolic algorithms used to solve linear and quadratic equations are different, but they are equivalent when discussed in terms of finding zeroes of functions. Moreover, these procedures are identical in a graphic setting, and they are structurally the same across symbolic and graphic representations. Properties of symmetry shared by quadratic and trigonometric functions provide another example, as do the

monotonicity of linear functions and the basic exponential and logarithmic functions. Finding extrema, general growth behavior, periodicity, domain, range, and intervals of increase and decrease are further examples of topics which can be explored using procedures in various representations, but which are equivalent in terms of the interpretations one can make from them across function classes. Hence, the ability to translate procedures across representations combined with a knowledge of various function classes yields a more general understanding of the conceptual result of a procedure and defines this stage of structural thought.

Stage 3: Procedural Networks as Permanent Constructs. The final stage of this process extends the student's ability to identify functional properties. Once functional properties are identified, the student can "see" a function as an object either with or without these functional properties. For example, a quadratic function could be viewed as a continuous function with exactly one extrema that is symmetric about a vertical line (with, of course, second-degree growth). Seeing a function as possessing these properties would allow the student to view the function as a well-defined object. Schwartz and Yerushalmy (1992) state that symbolic representations more often give rise to procedural conceptions, and graphic representations more often give rise to structural conceptions. This view of function requires an enormous amount of prior knowledge and familiarity with specific examples of functions and their properties, as well as advanced translation skills.

A Note on Terminology

Sfard refers to the process view as an "action" conception, but associates this with the process of computation. Dubinsky reserves the term "procedural" to describe, what is in my terminology, a more structural view of function (similar to the property-oriented View 2). Moschkovich, Schoenfeld, and Arcavi (1993) use the term "process perspective" to define a relational conception, where sets of inputs and outputs correspond (similar to the relational

View 1). Kieran (1992) uses the term "procedural" in a manner similar to Sfard and myself. It is quite evident that a universal terminology of these conceptions would be quite useful.

Synthesis of the Views

It should be noted that the previous views of function are not presented as a developmental stage theory of function. Instead, I hypothesize that although Views 1 and 2 may grow out of the process view of function, the two views may be independent of each other. It is quite possible that View 1 may be the first structural view of function that the student possesses, but it is also possible that it may be the last. Instruction is likely to influence the sequencing of this development.

These views of function are in line with instruction which makes use of symbolic, graphic, and numeric properties and representations. Because of the manner which graphing calculators can facilitate connections between different functional representations, incorporating them into the instructional process may benefit student development of a function concept image. Students who acquire a structural view can investigate functional contexts without a reliance on procedures and local function behaviors. Instead, emphasis can be placed on global behaviors such as growth rate or on relationships between specific local properties. Further, a structural view allows the student to better understand actions performed on a function, such as a "shift" translation (e.g., changing f(x) to f(x + 3)) or taking a derivative.

METHOD

Sample

This year-long study investigated the effects of graphing calculators on students' development of a function concept image. The study was conducted in two Algebra II classrooms at a private, mid-Atlantic high school which generally serves families of high socio-economic status. The graduating class of the sample consisted of 126 students, with PSAT scores above the norm (Verbal = 44.4 and Math = 48.5). The cumulative SAT scores of the graduating classes at

this school are normally around 1075, and 99% of the graduating class of 1993 entered a four-year college.

The two Algebra II classrooms were designated as Honors and consisted of students planning to take AP Calculus. The students were required to purchase a TI-81 graphing calculator which was used extensively in the instruction and course assignments.

Data were collected in the sections of the course taught by Lee Brook (see below, Instructor). Lee Brook (pseudonyms are used throughout) taught one section during the first semester and both sections during the second semester. Hence, year-long data were collected in one section of the course which contained 18 students. This group will be labeled the primary sample or primary section. During the second semester, data were collected in both sections which collectively contained a total of 36 students. It was concluded that the instructor approached each section quite similarly, using the same teaching strategies, materials, and often the same explanations. Although slight differences existed between the sections, I decided to include data from the secondary section in the analysis of student development wherever this seemed appropriate.

Case Study Participants. Using background information on the students provided by Lee Brook, as well as two preliminary classroom observations, three students were selected and agreed to participate in a series of interviews throughout the school year.

Erica. Erica was a sophomore who enrolled in the Algebra II course the previous year, but dropped midway through the first semester into an Algebra I course. During this time she gained one year's experience in working with the TI-81 graphing calculator. Erica received an A during the first semester, and a B+ during the final semester.

Curt. Curt was a freshmen who tested into the Honors Algebra II course. Because of this, he had no prior experience with the GC. However, he quickly learned the mechanics of the TI-81 and stated that he enjoyed working with it. Curt received a B+ for each semester.

Keith. Keith was a sophomore who transferred in at the beginning of the school year. Keith indicated to me that his previous schooling focussed heavily on procedures, did not use graphing calculators, and made little or no use of graphic representations. It did not take long for Keith to learn the mechanics of the GC, but he never seemed to "take to it" over the course of the year. Keith received a B and a B+ in the course during the two semesters.

Instructor. The instructor, Lee Brook, is an experienced classroom teacher and recipient of several teaching awards, including the Presidential Teaching Award. Lee Brook has 26 years of high school mathematics teaching experience, with the last 24 years at his current position. He received both the B.S. and M.A. in Mathematics, and occasionally teaches night courses at local colleges in addition to his high school duties. More detailed descriptions of Lee Brook, as well as the case study participants, appear in the next section.

Graphing Calculator. The graphing calculator used in this study was the Texas Instruments TI-81, one of the more affordable graphing calculators at the time this study was conducted. It has the ability to simultaneously graph four functions of the form $y = f(x)$ on a viewing screen approximately one inch by two inches. The TRACE feature places a cursor on a particular point on the graph and identifies the coordinates of that point at the bottom of the viewing screen. The user is allowed to move the cursor along the graph, and the coordinates of the points are continuously displayed. This feature allows the user to link the numeric and graphic representations in an immediate and dynamic fashion.

The user can alter the portion of the graph displayed by the GC through the use of the ZOOM feature. ZOOM changes the range of the x- and y-limits of the viewing rectangle, allowing the user to either ZOOM in on a small piece of the graph (by narrowing the range of the coordinates) or ZOOM out to capture a broader view of the function (by enlarging the range of the coordinates). Local functional behaviors (such as intercepts) can be found by ZOOMing in on the relevant portion of the graph and using the TRACE feature to identify the coordinates of the

desired point. Global functional properties (such as growth behavior and asymptotes) can be observed by ZOOMing out. The TI-81 also has programming capabilities, but these are limited by small memory capacity. The TI-81 is also capable of performing regression techniques, statistical analysis, and matrix manipulation.

Data Collection Procedures

Data on student development were collected through a series of written tests, individual interviews, and questionnaires conducted throughout the school year. The tests and interviews were included in three Unit Investigations (UI), and were conducted shortly before and after each unit's instruction. The UIs, approximately two weeks in length, consisted of three different classes of functions (linear, polynomial, and exponential). The linear unit occurred during early October, the polynomial unit during February, and the exponential unit took place in May. Hence, every student in the sample participated in UI#2 and UI#3, while each student in the primary sample participated in all three Unit Investigations.

Each written test required the student to define function, discuss "what the idea of function" meant to them, and to provide an example. Some test items dealt with skills related to specific properties of the unit's function class, while other items allowed for investigations of more than one property. The students were also asked to make connections between tasks written in different representations, and a quiz given between UI#2 and UI#3 dealt with this as well.

Student interviews began with tasks in which students' could exhibit their strategies when translating across functional representations (see Figure 2). This was followed by problem solving tasks designed to determine the case study participants' structural conceptions of function, as well as uses of the graphing calculator. For example, tasks involving maxima were given in different functional representations and classes, and the students were asked to discuss commonalities in their solutions. The participants were also given function recognition

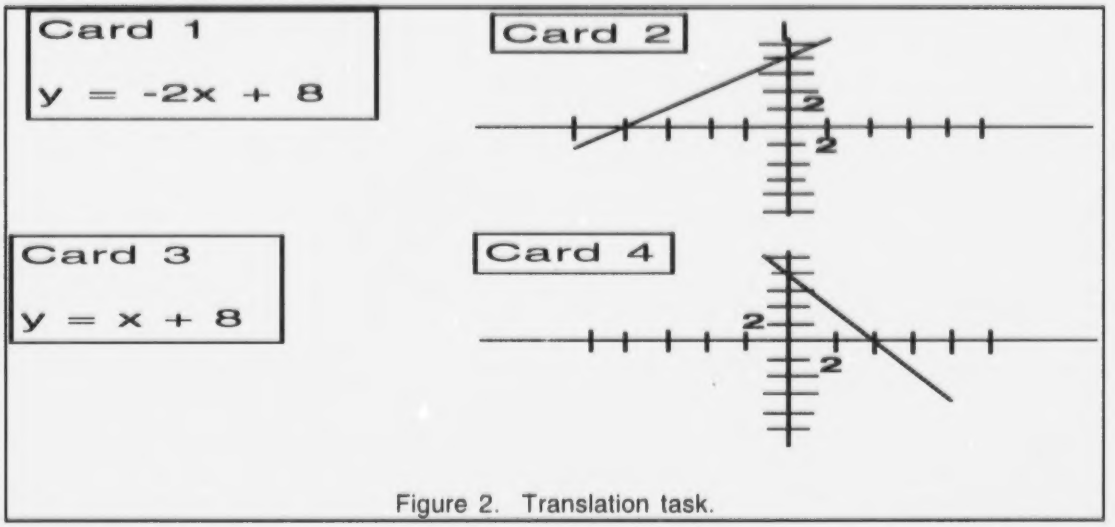


Figure 2. Translation task.

tasks to help determine their View 1 structural conceptions. Each interview ended with a discussion of function or use of the GC.

Class-wide questionnaires were given outside of the Unit Investigations, during both November and May, which dealt with uses of the graphing calculator and the concept of function. The GC Questionnaire focussed on determining the types of tasks on which students would most likely use the graphing calculator. The Function Questionnaire determined if the students agreed more with a process or relational view of function. Periodic classroom observations were conducted weekly and provided descriptive information on the instructional process.

Analyses focussed on determining the students' concept image of function. Since function View 2 is dependent on generalizations across function classes, student work was examined across the year. This analysis also allowed for observing changes in students' conceptions of function.

RESULTS

Although this paper focusses on evidence relative to learning outcomes, a brief overview of the instructional process from this study will now be given. The GC provided the instructor a

means of quickly changing symbolic function parameters in order to better discuss global functional properties of a given function class. For example, simultaneously graphing the functions $f(x) = 2^x$ and $f(x) = .5^x$ allowed for discussions of exponential increase and decrease. The TRACE feature allowed these graphic representations to be quickly tied to numeric ones, and the instructor was careful to relate the symbolic expressions to these as well. Hence, the GC facilitated explorations from a multi-representational viewpoint. The need for the symbolic expression when graphing a function on the GC helped to "force" these connections into the lesson. The instructor used this to his advantage throughout the year, but he increased his use of the GC as the year progressed and as more novel topics and function classes were being introduced. The GC also allowed the instructor to explore textbook problems in his own fashion, changing them into explorations which were more pertinent to the lives of the students. For example, the ability to quickly change the symbolic parameter of a linear function allowed for investigations into the operating costs of cars with varying gas mileage.

Development of a Structural View of Function

Six times throughout the year each student in the sample was asked for their thoughts on the concept of function, once before and after each of the three Unit Investigations. Specifically, the following questions were asked:

1. *(Definition item) Give a precise definition of function.*
2. *(Discussion item) In your own words, relate what the idea of function means to you.*
3. *(Example) Give an example of a function.*

The students' initial conceptions of function primarily dealt with specific productions of outputs, usually from a symbolic formula (see Tables 1 and 2). This result is in line with prior research (Sfard, 1991) and we have previously referred to this as a process view of function.

Insert Tables 1 and 2 About Here

Tables 1 and 2 indicate that students' formal definitions of function contained more references to a relational view of function than did their informal discussions of function. For example, on the posttest of UI#1, CK wrote the following when answering these items:

1. (Definition) A function is a relationship where each input has one output. The domain of a function is the input, the range is the output.
2. (Discussion) A function is where you substitute a number for a variable and get one answer.

CK's definition was written in relational terms, but he had procedural thoughts on the nature of function since they dealt with the computing of specific outputs. On the pretest of UI#2, LJ wrote the following (both were classified as relational responses):

1. Any relation where for every input there is only one output.
2. An equation where for every x there is only one y.

Other students, such as MR, gave a similar response for the Definition item, but for the Discussion item used numeric terms such as "Every number you put into this equation you will get back only one number". Because of the direct reference to the production of specific outputs, this response was classified procedural.

Throughout the year, many other students defined function and personalized a discussion of function in a similar fashion as CK. Relational terms were used in the definition, but procedural terms were used to discuss what it meant. The fact that the rise in relational terms is sharpest during UI#1 is due to the fact that this is where Lee Brook first gave a formal definition of function, stating, "It's a relationship, it's not just sticking in an *x*". Lee Brook followed this with a toll bridge metaphor, analogous to the more common function machine. Student responses appeared to be framed in part by this instruction, even though rote knowledge of the definition was never stressed. Students used terms such as "input", "output", and "relationship" to define function, but the intended meanings of many of these definitions involved only the production of single outputs and not the connection between domain and range, or even connections between specific elements of the domain and range (ordered pairs). In

addition, by the end of the year, 22 out of 34 students defined function without the use of a functional representation, but instead used terms such as "inputs and outputs". However, 21 out of 34 students used symbolic language in discussions of function. All but one example of function given by the students throughout the entire year was in symbolic form, and most were examples of linear expressions.

These data indicate that the students increased their awareness of the relational properties of function over the course of the year, but to a limited degree. This evidence is supported by the Function Questionnaire items in Table 3 which illustrate the procedural view of function the students carried with them throughout the year, as 85% of the students continued to agree more with function as a computation than as a correspondence between sets.

Insert Table 3 About Here

To further illustrate, let us consider the comments of Erica, one of the case study participants.

Pre-Task Interview #1
DS: What do you think of when you hear the word function?
Erica: Um, I think I think of y, y values. Um, f(x), and kind of dealing with graphing.

Post-Task Interview #2
DS: Can you tell me what a function is.
Erica: Well, its an equation in which certain numbers are put into, um, which will give you one answer overall. Each input can produce one and only one output.
DS: Ok, that's a good definition, but beyond the definition, how do you think about what a function is? Is there anything besides just that definition?
Erica: Um, (long pause) I guess just like, um, an equation that can be applied to numbers.

Post-Task Interview #3
DS: Part A (Definition item) on the written tests was always "state the definition of function", and the second part was "state what the idea of function means to you". Did those mean different things to you?
Erica: To me they were pretty much the same. You have the technical definition, but mine means the same, just in easier terms to comprehend I guess.
DS: OK, do you remember the "in your words", what you said for that?
Erica: (pause) Um, I think it was something like, um, for any number put in the equation there can only be one answer that develops.

These comments indicate the procedural manner in which Erica's definition of "each input can produce one and only one output" should be interpreted, and how this changed very little over the course of the year. Keith made similar comments at the end of UI#3.

Curt seemed to develop his concept image further than Erica and Keith, with the graphic representation playing a major role in his development.

Pre-Task Interview #1
DS: In your own words, what comes to mind when you hear the word function?
Curt: I just think of a problem, like $4y + 2x = 9$, something like that.
DS: Ok, when you say a problem, um, the $4y + 2x = 9$ is the problem?
Curt: Yeah, but I'm not too sure what a function is.

Post-Task Interview #2
DS: Ok, if I asked you what a function is, what would you tell me?
Curt: Well, its uh, uh (pause) in a function (pause) every, um, I'm terrible at this, uh, graph of, um I really don't know. (pause) It's the relation of something to something, the numbers in a relation.
DS: Ok, instead of trying to come up with a formal definition, talk about it in other ways. Can you talk about what you maybe think a function is besides trying to come up with a formal definition?
Curt: Well, its using like, data that you have to get the graph of something for. Like if I just had an equation for something, an equation that would give me a maximum, but unless I would graph it, I don't think it really gives me anything. So its just a relation of numbers that if, you know, use graphically, or if you could just figure out by finding an equation what you're highest is or whatever it is you need to figure out. What you're best profit is going to be, what you're best average is going to be, or whatever.

Post-Task Interview #3
Curt: Functions are . . . the key to everything that we've done this year. Well, they seem that way. I relate it to everything, well in math, I don't relate it to history (laughs). I relate a function to, when I'm looking at a graph it's now a function. Like, before if it was a problem, I'd find it's solution is a function. The graph of the function is involved. So my view of function has become more broad, and I think its the basis of everything we've done this year.

Curt's concept image of function began as "a problem" to be solved, clearly a procedural view. It evolved to include relational aspects and assistance in application tasks, and culminated in a truly multi-representational view ("when I'm looking at a graph its now a function") which is connected to numerous aspects of algebra. The graphic representation, and the functional properties it helped to make explicit, appeared pivotal in Curt's development. Curt's definition

also reflected an emphasis on problem solving which was consistent with the instruction of Lee Brook.

Since only 12 of 34 students discussed function in the same relational terms as Curt on the Discussion item of the posttest in UI#3 (Table 2), generalizations to the entire class cannot be made. In addition, the second time the Function Questionnaire was given, Curt was 1 of only 5 students who agreed more with the correspondence view of function than the computational view of function. These data indicate that the instruction was lacking in regard to the development of a View 1 structural conception. Although the graphic images of the GC provide opportunities for discussions of the relational aspects of function, these were not emphasized by the instructor throughout the year.

Task analyses of the case study participants provided more in-depth information in regard to the attainment of a View 1 conception. Tasks were given during the Post-Interview of UI#2 which consisted of 12 sets of cards depicting two examples and non-examples of function in each of the numeric, graphic, and symbolic representations. The participants received no formal instruction on the majority of the function examples in these tasks (they were not the elementary functions common to the course), and all but one of the graphs were discontinuous.

All three of the case study participants provided accurate reasons for the numeric examples and non-examples of functions. All of the participants successfully identified each of the symbolic examples with the exception of Card 8 ($x^2 + y^2 = 16$), although somewhat faulty reasoning was used on some. For example, Erica and Keith indicated that $x^2 + y^2 = 16$ was a function, stating that it could be "solved for y". The GC requires an expression of the form $y = f(x)$ in order to produce a graph, and this may have been a source of the students' thinking. This reason was used by Erica and Keith for all four symbolic examples. Here, Erica reveals her thinking on Card 1 ($x = y^2 - 2y + 5$):

Erica: No because it would have to be y by itself, I guess you could get it by itself.
DS: Ok, you mean y= something?
Erica: Yeah.
DS: Ok, suppose you couldn't get y= something, then is it a function?
Erica: Then (pause) um, it wouldn't be a function.

Only Curt stated that the two non-examples, Cards 1 and 8, would violate the univalence criteria. Keith was the only participant to indicate that Card 1 could be interpreted as a function $x = f(y)$.

Explanations on the graphic examples were much more diverse. All three participants were fooled by graphic properties and behaviors not associated with the elementary functions used in the course. Both examples of function were not identified by all three participants, as they were misled by the finiteness of the graph or the discontinuity. On Card 5 (see Figure 3) Erica stated:

Erica: It's not a function.
DS: How come?
Erica: It has to be continuous (pause).
DS: If I connected these points would that be a function?
Erica: No, you can't really get an equation that would give you that. It's more like a slope.
DS: Ok, so is the fact that its not continuous or that you can't get an equation . . .
Erica: Both.

On that same card, Curt stated:

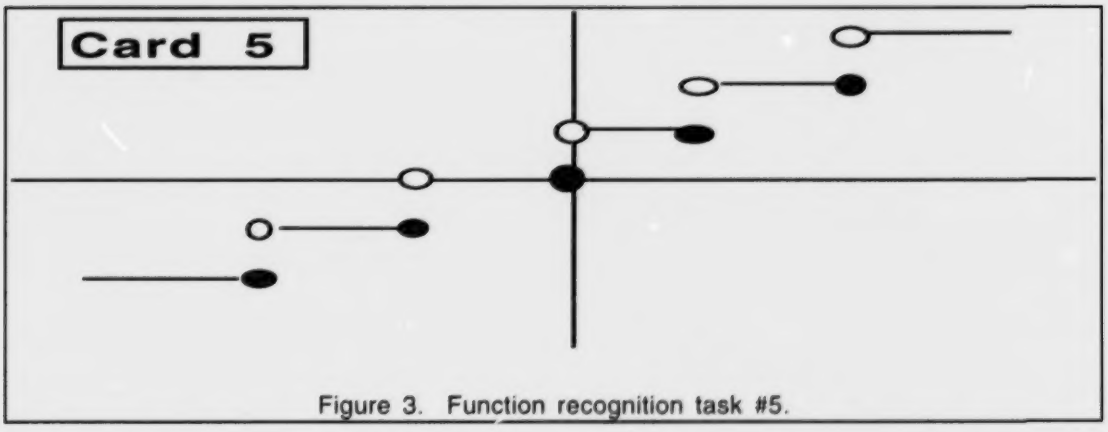


Figure 3. Function recognition task #5.

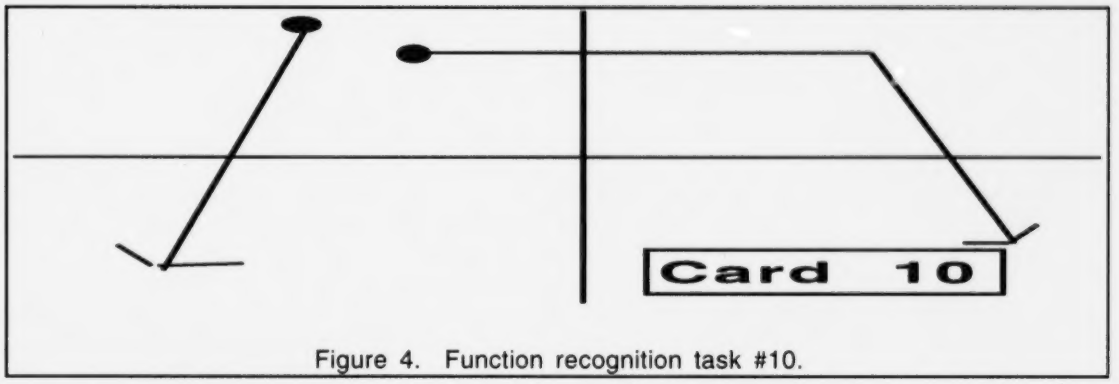


Figure 4. Function recognition task #10.

Curt: This one, um (pause) from what I know, I guess, these continue out right?
DS: No, that's where it stops.
Curt: Oh, well then no, because they don't continue out.
DS: Ok, what if they did continue out?
Curt: Well, if it did continue out I'd say technically I guess, well, maybe, I'm just guessing bu[illegible] from what I know I guess it would be a function, but it doesn't really look like one at all.
DS: Why do you say it is?
Curt: Well, because there is one y-value for every x-value, but I'm not sure if there has to be a y for every x. It doesn't really look like a function, but it fits the requirements I know of.

Curt seemed to be doubting his own feelings due to the unusual behavior of the graph. By this time his written definition of function was "exactly one output for each input", but his last statement in this passage seems to indicate confusion on what this really meant. The "gap" in the graph of Card 10 (see Figure 4) caused similar confusion:

Curt: I would say (it's not a function) because there's not a y-value for every x.
DS: If that was connected would it be.
Curt: Um (pause) I don't think so, well, maybe. Um, the only question is the sharp turn, but I would think so, if it was connected (it would be a function).

The students' concept image of the graph of a function clearly involved continuous functions with infinite domains which can be represented by a symbolic expression $y = f(x)$. The GC requires a symbolic expression, solved for y, in order to produce a graphic representation. Erica's reasoning that "you can't really get an equation" indicates that the requirement of a symbolic representation is a misconception the GC can foster. In addition, the graphic images that the students normally viewed on their GC did not resemble the functions on

these cards, as the instruction predominantly dealt with elementary functions. This also may have led to the other misconceptions involving the discontinuities of the graphic examples.

Discussion. The students in this study, like those of Sfard (1991), appeared to have difficulty generalizing a process view of function into an object-oriented relation which dealt more with set-based conceptions. The evidence indicated that most students were able to define function as a relationship between sets, but this conception was less prevalent when discussing what this meant. In other words, the students were able to produce a set-based, relational definition of function, but these conceptions were less prevalent in discussions or problem solving situations. The exclusive use of elementary functions when working with the graphing calculator may have hindered this development, as it appeared to foster misconceptions in regard to the nature of graphic functional relations. We now examine if a property-oriented View 2 structural conception was achieved.

Structural View 2

One way in which students might have developed structural View 2 would have been to focus on the type of function (linear, exponential, etc.) and relate this to various graphic, numeric, or symbolic features. Since the GC allows for problem solving strategies in each of these representations, incorporating it into the instruction would seem to facilitate a View 2 conception.

The written tests of UI#2 and UI#3 contained tasks which asked the students to analyze a function given in symbolic, numeric, or graphic form. The tests of UI#2 were designed to focus more on the use of the GC for tasks written symbolically, and UI#3 dealt more with differences in problem solving strategies amongst tasks written in different representations. Every student was required to indicate whether or not they used the GC in the solution process and to discuss the manner in which they solved a particular task. Questionnaire items were also used to determine the level of GC use of the class.

GC Questionnaire. Responses to the GC Questionnaire indicated that the students made extensive use of the GC for tasks which asked them to produce or analyze a graph (see Table 4). The students increased their tendency to initiate the use of the GC and graphical analysis from 59% to 85% over the course of the year, and similar increases were found in the students' view of the necessity of the GC in the solution process. Hence, the students indicated that they increased their use of the GC the most on tasks which implicitly called for graphical analysis. Solving the expression $x^2-5x+4=0$ graphically would illustrate this problem solving behavior. These data indicate that the GC had an effect on the students' problem solving tendencies, and it may have had an effect on their property-oriented view of function. To help determine the latter, we now describe specific uses of the GC over the final two UIs.

Insert Tables 4 and 5 About Here

Unit Investigation #2. The students increased their use of the GC following the instruction of UI#2 (see Table 5), particularly on items which called for finding zeroes of functions. This supports the data from the GC Questionnaire in that increases were greatest for tasks involving graphical analysis. A closer examination of the students' strategies indicated that they made great use of the ZOOM and TRACE features of the GC when using it in the problem solving process. They also combined symbolic, graphic, and numeric techniques to find solutions. For example, on the task "Find the zeroes of $f(x) = x^3 - 19x + 30$" from the posttest of UI#2, 31 out of 32 students indicated that they used the GC to solve the problem. The following strategies were given:

JG: I graphed it and zoomed in and checked by substituting the numbers in.
MM: I graphed the function, listed the possible zeros of the function, and picked -5. I evaluated the function for -5 on my calculator, and it equalled zero. Then I factored (x+5) out of the function using synthetic division. I factored the resulting quadratic function to get the other two zeroes.

CR: I Traced to find the place where the graph crossed the x-axis.
BC: Boom, boom, boom and zoom, zoom (Graph and ZOOM).
MD: I found possible rational solutions, graphed it, narrowed the solutions down, and plugged them in.

All of the students' strategies reflected one of the above, with ZOOMing in on the x-axis of a graph being the most popular. Note that MM and MD made use of the Rational Root Theorem, a strategy Lee Brook used in class prior to introducing graphic techniques. Multiple functional representations were also used in their solutions -- the fact that the polynomial has three zeroes came from the degree of the expression, the possible zeroes came from the Rational Root Theorem, and the answers came from TRACing the graph on the GC and using the possible rational roots as a guide.

Some students (37%) still preferred symbolic strategies when solving relatively simple tasks like "Find the zeroes of $f(x) = x^2 - 4x - 32$" (see Table 5). Two of the students who did not use a graph stated that they used the quadratic formula, while the rest "simply factored it". Even though the success rate stayed virtually the same on this item, the number of students who switched from symbolic to graphic methods increased considerably.

Insert Table 6 About Here

On a task which was possibly worded toward symbolic methods, "Factor $f(x) = x^3 + 3x^2 - 4x - 12$", 41% of the students solved it with the GC by graphically finding the zeroes of the function (see Table 5). Table 6 indicates that this task was by far the most missed item on the written tests of UI#2. However, on the pretest, 8 of the 11 students who used the GC in the solution of the problem were correct, and 13 out of 13 students were correct on the posttest. BL stated, "I found the zeroes on the calculator and used them to factor down the equation." Another item, "Find all values of x such that $5x^2 + 9x - 3 > 4x + 5x^2$", also had very low usage

of the GC in its solution. The success rate for those who did and did not use the GC were similar, with GC users correctly answering 3 out of 5 on the pretest and 7 out of 8 on the posttest.

The above results support the findings from the GC Questionnaire on students' problem solving tendencies with the GC. The students preferred to use the GC on most problems, but still

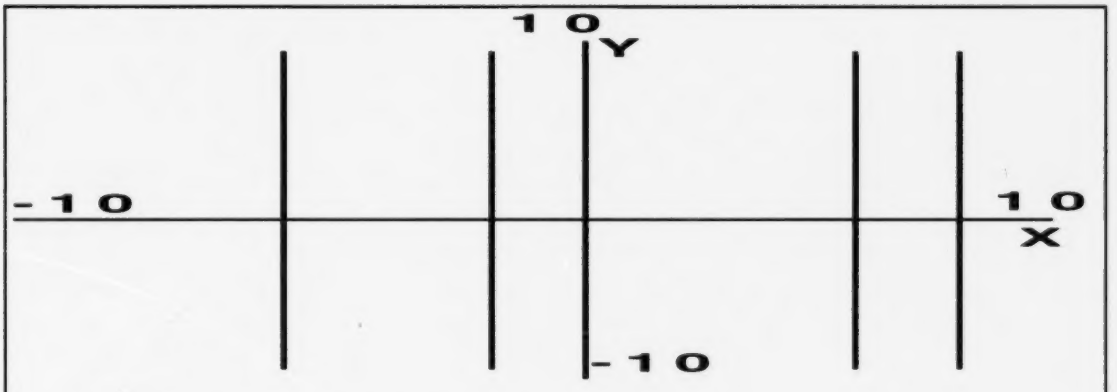


Figure 5. Graph of *f(x)= -x5 +12x4 +10x3 -120x2 -9x +108* using the standard viewing window.

were capable of solutions which did not make use of graphic representations. The factoring item suggests that the students were, at times, more able to find a correct solution when using the GC.

This accuracy also held true, in part, on the graph construction items (see Table 7). I designed functions in which significant portions of the graph occurred outside of the default viewing window. If using the standard viewing window, both graphs of UI#2 would appear as disconnected vertical lines (see Figure 5). This could only be corrected if changes were made to the range values of the y-coordinates. In addition, item 1 contained an extrema and an x-intercept outside of the [-10,10] range for the x-coordinates, forcing changes along this axis as well. The students were forced to use their knowledge of the behavior of the function class in order to recognize that the default graph was incomplete.

Insert Table 7 About Here

Table 7 indicates that most students were capable of recognizing the need to extend the range of the y-values in order to display a continuous graph, but less capable of investigating graphic behavior outside of the default range of the x-coordinates. This is likely due to the fact that the students initially saw the graph in Figure 5, understood that the "lines" were merely parts of the graph which eventually must connect, and changed the range of the y-coordinates accordingly. Of the students who expanded the range of the y-coordinates, less than 15% still drew a discontinuous graph for item 1 on the pretest. No one on the posttest drew a discontinuous graph for item 2. This indicates that the students were successful in investigating graphic behaviors outside of the default range for the y-coordinates. However, on the posttest, two-thirds of the students continued to fail to investigate the graph of item 1 outside of the [-10,10] range for the x-coordinates. This is despite receiving instruction on the number of intercepts a fifth-degree polynomial should have. It also should be mentioned that approximately one-third of the students who included all five intercepts in their graph did so in a discontinuous manner. Therefore, the number of discontinuous graphs may have been higher if more students had included all of the intercepts in their graph of item 1. Hence, when constructing a graph, students were much more attentive to a function's continuity than the number of zeroes it may have. It also may be true that students are more likely to examine functions vertically and investigate behaviors for large values of the range (y-coordinates), and less likely to expand a graphic image horizontally to investigate behaviors for large values of the domain (x-coordinates).

Unit Investigation #3. Responses on the written tests in UI#3 indicate the flexibility in the students' problem solving strategies. They also may indicate that the students did not always

view the GC as essential for success, as only minor increases were found in the use of the GC on the written tests of UI#3 across the unit. It should first be noted that these items were written in symbolic, graphic, or numeric representations. In addition, unlike the tasks of UI#2, some of these items were written in narrative form.

The students had very little trouble on the task which required the graphic evaluation of a function. The students initially had the greatest difficulty with the symbolic task involving compound interest, but showed great improvement after instruction on exponential and logarithmic functions. Success on this item increased from 21% to 83%, and only a small increase was found in the number of students who used the GC on this item (see Table 8).

Insert Table 8 About Here

On the numeric item involving exponential growth, 12 students used linear regression techniques on the pretest for this exponential situation, and 6 did so on the posttest. These students applied their prior problem solving procedures from the linear unit and programmed the GC to find the line of best fit for the data. The fact that 6 students continued to use linear regression <u>after</u> instruction on exponential functions indicates that some students may have difficulty extending functional ideas beyond linearity for numeric problem solving settings. It also provides an example where problem solving techniques offered by the GC can be confusing, particularly since exactly as many people used exponential regression techniques as who inappropriately used linear regression. The other students who successfully answered this item numerically analyzed the changes in the y-coordinates to discover the pattern in which they increased.

The graph construction item in UI#3 involved less complicated behavior than those in UI#2, but one-half of the students on both the pre- and posttest misinterpreted the graphic

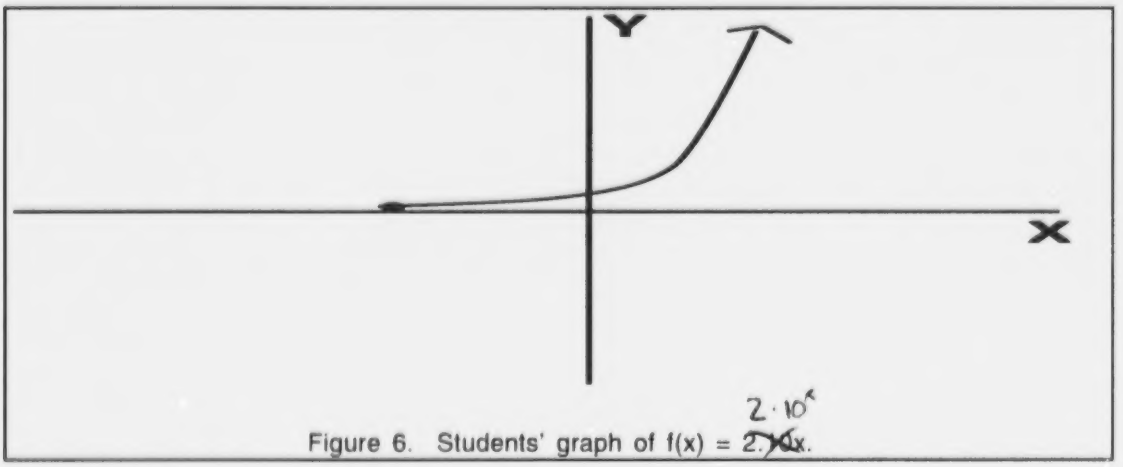


Figure 6. Students' graph of f(x) = $2 \cdot 10^{x}$.

image of the GC as indicating that the function's domain began at a fixed negative value of the x-coordinates, instead of continuing out to infinitely negative values (see Figure 6).

Summary. The students' extensive use of the GC in problem solving situations indicates that it did have an effect on their property-oriented conceptions, but the decreased use during UI#3 shows that this effect may be limited. Data from the GC Questionnaire and written tests of UI#2 and UI#3 provide evidence that the GC was integral in the problem solving processes of the students.

The above evidence suggests that the students had more of a multi-representational view of function than the initial definition and discussion items suggested. Also, the View 1 conception which dominated the definition and discussion items was almost non-existent in these problem solving situations. The GC did cause misconceptions to form, including incorrect assumptions about continuity and restricted domains. To further investigate the formation of a View 2 structural conception, we now examine the translation strategies of the students.

Effect of the Graphing Calculator on Representational Translations

Throughout the year, the students were asked to solve tasks designed to measure the degree to which they could relate different functional representations. For example, a quiz

given between UI#2 and UI#3 asked the students to discuss any commonalities in their solutions to tasks involving zeroes of functions which were written in three different functional representations. 62% of the students explicitly mentioned the structural similarity of these tasks. LJ stated:

LJ: All of the questions asked for the zeroes of the equation. The solutions were, where on a graph the line crosses the x-axis. You could put a graph to all of the problems in order to find the zeroes. Two of the solutions were estimations.

Note the use of the graphic representation to describe similarities in the tasks, something common to many of the responses. These comments indicate that the students' concept images were often in multiple representations, which allowed the students flexibility in choosing an appropriate problem solving strategy. LJ's emphasis on the graphic representation suggests that he made this his "anchor" representation, and his ability to equate procedures across function classes and representations is evidence of an evolving View 2 structural conception.

During the Pretask Interview of UI#2, the case study participants solved tasks in symbolic, numeric, and graphic representations, all involving maximization. When asked to discuss similarities in these tasks, Erica and Curt recognized the similarity in the goal of the tasks (maximization) despite the different representations, and discussed this using graphic language.

Erica: Oh, each problem involves a vertex I guess, and it represents the maximum point of each. Like on this one you wanted the vertex itself, and this one the maximum profit, and this one wanted his best scoring season, so uh, the vertex that way. Um, these two are parabolas, but the profit one I'm not too sure about.

Curt went a bit further discussing the use of a graphic representation:

Curt: If I just had an equation for what Jimmy Jumpshot plays, how his scoring went, an equation that would give me a maximum, but unless I would graph it, I don't think it really gives me anything.

It appears that Erica and Curt relied heavily on the graphic representation in problem solving, as well as in making connections between representations. Like LJ, they used the graph as an anchor with which to think about the maximizing characteristics of the tasks.

The most extensive information on translation strategies involved tasks given to the case study participants before and after each of the three Unit Investigations. The tasks involved cards containing functions in different representations, and the students were asked to determine any similarities or differences in the functions (see Figure 2). Attention was given to the manner in which they related the various representations of a given function. Only minor differences were found in regard to the success at which each of the students translated between two particular representations of the same function, with a slightly higher success rate for translations involving symbolic equations (E) and numeric tables of values (T). No major differences were found in the success rates across Unit Investigations, but all three students were more able to correctly translate between each pair of functional representations after the instruction in each UI (with the exception of Keith during UI#2).

Of greater interest is the manner in which these translations were made. Generally, students either translated on a local, point-by-point basis or used global properties and behaviors. A local translation strategy would involve the student checking several values of the domain on each representation and determining if they have identical range values. This approach would correspond to a procedural view of function. A behavior-oriented strategy would discuss things such as "the same slope and intercepts" or "both grow exponentially in the same way" and would correspond to a more structural view of function. Sometimes the students would combine these strategies. For example, when translating between two representations of a linear function, a student may notice that each has the same y-intercept (without evaluating a function or plotting a point) and then check "any other point".

An analysis of these tasks revealed that the students were much more inclined to use local "point-checking" strategies when relating Equation-Numeric Table (ET) representational pairs, and more likely to use global properties when relating Graphic-Equation (GE) pairs. Erica used local strategies more often when translating between Graph-Numeric Table (GT) pairs, while no pattern was found in the strategies of Curt and Keith on these tasks. A summary of the observed translation strategies of Keith can be found in Table 9. Since the GC requires a symbolic expression in order to produce a graph, this may have facilitated the advanced translation strategies between these representations. In any event, the fact that the translation strategies were most advanced for these representational pairings is further evidence of the positive influence of the graphic representation on a View 2 structural conception.

The strategies of Curt and Erica went from local to global after the instruction in UI#1. Similar changes during the other Unit Investigations were not as noticeable. Several of the local strategies involved the use of intercepts, especially on GE and GT pairings of linear functions. Very little difference was found in the translation strategies of the students for different function classes, as their strategies stayed consistent throughout the year. Hence, the functional representation appeared to be more of a factor than the function class in determining the strategy used.

The following comments of Erica during UI#1 help to illustrate the advancement in the students' translation strategies. During the Pretask Interview of UI#1, a task involved the graph and equation of the functions $y = x + 8$ and $y = -2x + 8$ (see Figure 2):

Erica: (long pause)
DS: Which ones are you comparing now.
Erica: Cards 2 and 1, they both have the y-intercept the same, and it appears that would be the solution, uh, yeah, uh (pause) Card 3 has the same intercept too, um, (pause) oh Card 3 is the solution to Card 2, um (pause) Card 1 wouldn't be but it does have the same y-intercept . . . well all of them share the same y-intercept as Card 2. Eight is the y-intercept, um (pause) and Card 1 would be the solution to Card 4.
DS: How are you comparing these?
Erica: Um, well its kind of hard to see, but I was just putting numbers in.

DS: Ok, which numbers?
Erica: For Cards 1 and 4 I guess I put zero in for y and it would have to be four, which is what the x is on here.
DS: OK, in general, any similarities or differences?
Erica: Just that the y-intercept was the same with all of them.
DS: OK.

We see evidence that Erica is concerned with important aspects of the linear function, namely the intercepts, but her translations rely on point-checking strategies. After the instruction in UI#1, Erica faced a similar task. Here, Cards 2 and 3 relate the graph and equation of the function $y = x + 3$, while cards 4 and 5 represent the numeric and symbolic representations of $x-y= -1$:

Erica: (pause) Cards 3 and 2, um, Card 3 would be the equation of the line.
DS: How did you know that one?
Erica: Um, because three is the y-intercept of the equation and it runs through three, and then, um, if the slope is just one the line would intercept at negative three.
DS: Ok.
Erica: (pause) Cards 5 and 4, the x and y's fit into the equation.
DS: How did you check that?
Erica: I just plugged the numbers in.

Here we see how Erica is beginning to use the property of slope to make connections between the representations. However, she used slope to check the point (-3,0), not as a defining property of the function. We also notice how she used local properties on the ET pairing. Erica's strategies continued to get more sophisticated, as evidenced by the following task given before UI#2. Cards 3 and 4 are the graphic and symbolic representations of the function $y = -2x + 4$:

Erica: Card 4 is the equation for Card 3, they both have y-intercept of four and slope is -2.

Here we see that Erica is satisfied that both cards represent the same function without any reliance on point-plotting or evaluation. It also shows that she is beginning to objectify linear functions, as they are completely determined by the two properties of slope and y-intercept.

Curt showed similar development during UI#2 in regard to polynomials. In this task, Cards 1, 2, and 4 represent either the graph or equation of a fourth-degree polynomial, and Card 3 is the equation of a third-degree polynomial:

Curt: Um, let's see, I'd have to say that I don't think Card 3 has anything to do with the other ones because I think it has different humps, I don't know. Cards 1 and 2, well obviously they aren't related so I'll try, um, Cards 4 and 2. Um, I don't know about any of these.
DS: Can you make any general comments?
Curt: Just that, (pause) no (laughs).
DS: Ok, how come you think you had so much trouble with those?
Curt: Because I really, there's no way I could figure them out, I mean there's no coordinates or anything.

Although it appears that Curt is somewhat familiar with the behavior of third- and fourth-degree polynomials, this final comment indicates that he is still dependent on sets of ordered pairs and point-plotting strategies when translating between different representations of this function class. This dependency waned by the end of UI#2. Here, Curt indicates that Cards 2 and 3 represent the graph and equation of different third-degree polynomials:

Curt: Uh, Card 2, well let's see, um, I don't see any similarities other than Card 3 could be the graph of a third-degree equation, but it's going from positive to negative so it couldn't be the graph of Card 2 ... Well, Cards 2 and 3, if you stuck a negative in front of the x^3 it might work.

Curt also used global strategies during this interview on a GT pair:

Curt: Ok, well, looking at Cards 2 and 3, Card 2 could be the graph of Card 3 because looking at it, the point (0,0) in Card 2 is a point as it is in Card 3, and both of them look to be symmetrical over the y-axis. Yeah, both of them, and also, except for zero, all the points are negative on both, it goes from negative to negative and never crosses the x-axis, so the y's are positive.

Here Curt initially focussed on a single point, but then used symmetry, the position of the graph below the x-axis, and the general shape of the graph (starting and ending "at" negative infinity) to translate between representations. These data illustrate the effects of the previously discussed explorations with the graphic images of the GC on Curt's conceptions of function.

During the final interview, the participants were asked to discuss the methods in which they made translations. By the end of the year, Curt stated that he was more comfortable with pairs involving graphic representations due to his ability to recognize and relate growth behavior.

Curt: Some of the tougher ones I've had to relate them, you know, where it was tougher to plug things into stuff.
DS: Were those tougher when you couldn't plug in?
Curt: Well, it would be tougher to figure out an exact answer, but in general it would be just as easy.
DS: Ok, how about a graph and a corresponding table of values, talk about how you related those.
Curt: Well there, ok, that's easy, because you can just tell, the first thing I would look at is the direction of the graph from the values, so they're related that way. I mean, because with the steepness that I was just looking at, and then with the graph you learn alot from the direction it goes, and the values, as opposed to an equation, you can really see more which direction it goes.
DS: Ok, how about the equation and a table of values, how did you relate them?
Curt: They're a little tougher to relate for myself, but its kind of easy. Well, you get the idea from working with the equation, but they're the toughest because with the graph, saying its related, well you're looking at a picture, but these two when you say they are related its because the values are corresponding, you have to do work other than use your eyes to decide.

Here, Curt explicitly stated his use of local translation strategies on ET pairings, and how by the end of the year this was less efficient than concentrating on global, property-oriented strategies which he commonly used on pairs involving a graphic representation.

It appears that the graphic representation most often gave rise to structural translation strategies, and the numeric representation most often gave rise to procedural translation strategies. This indicates that graphs are vital in the students' advancement towards functional View 2. A deep understanding of functional properties, and the objectification of function as an object containing these properties, would seem to be highly dependent on the graphic representation. The manner in which the GC can facilitate graphic investigations appears to be beneficial to this type of understanding.

DISCUSSION

Curriculum planners have little evidence on which to base decisions regarding the use of the graphing calculator in algebra courses. This study has attempted to address this need by providing information on the learning outcomes in a high school Honors Algebra II course which made extensive use of the graphing calculator.

Students' Conceptions of Function

The students' ability to objectify function was examined at two distinct levels; 1) as a causal and dependency relationship between two sets (View 1), and 2) as a connected network of functional properties in a variety of classes and representations (View 2). Many of the students failed to objectify the concept of function from a relational perspective, although the written definitions they provided would suggest otherwise. The students also did not appear to objectify the concept of function beyond quadratics in terms of functional properties. Specific function classes were objectified, but the general notion of function as an object possessing a variety of properties was not made. However, evidence existed that the students were beginning to deepen their property-oriented conceptions after the instruction on exponential functions at the end of the year, and continuous rises in a relational conception were also seen.

The extensive use of the graphic representation provided by the GC enabled the students to more readily focus on global function properties, thereby fostering the development of a View 2 structural conception. The strongest evidence for this came from analysis of the strategies used by the case study participants to translate between functional representations. Pairs involving graphs allowed for global translation strategies, and those which involved a symbolic representation led more to local strategies. Evidence indicated that the case study participants obtained procedural translation ability (PTA) for all classes of functions, but acquired structural translation ability (STA) only for linear, and possibly quadratic, functions. Procedural translation ability allows one to move the function definition across symbol systems.

Graphing a function given its equation, writing the equation of a line given two points, and graphically finding zeroes of functions are examples of PTA. Structural translation ability involves the relating of meanings and functional properties across symbol systems. Numerically understanding why f(x)=1/(x-1) has a vertical asymptote at x=1, how the numeric value of the slope relates to its graph, and why the graphs of f(x) and f(x+3) are similar illustrate examples of STA. The fact that the case study participants predominantly used point checking strategies on all but the EG pairings indicates limited STA, as properties had to be translated one point at a time. In other words, general properties had to be viewed on a very local level (point-wise) in one representation, translated, then generalized in this other representation in order for the meaning to be sent across the symbol systems (if meaning was sent across at all). These data also provide further support of the positive effect of the graphic representation on STA, as well as on the general understanding of function.

The manner in which the students defined function was somewhat inconsistent with the manner in which it was discussed, both in terms of representation and emphasis on relational properties. The students were able to define function in a structural manner, but informal discussions revealed that these conceptions developed to a more limited degree. Whereas evidence regarding property-oriented conceptions came strictly from an analysis of problem solving tasks, evidence on a relational view of function came mainly from analysis of the students' written definitions and discussions of function. These multiple data sources allowed for a broader view of the total concept image, and future studies of students' conceptions of function would benefit from this type of analysis (see Figure 7).

Effect of the GC on Learning

Student responses to the GC Questionnaire and written tests provided strong evidence that the students relied on the GC in numerous problem solving situations. These tendencies had

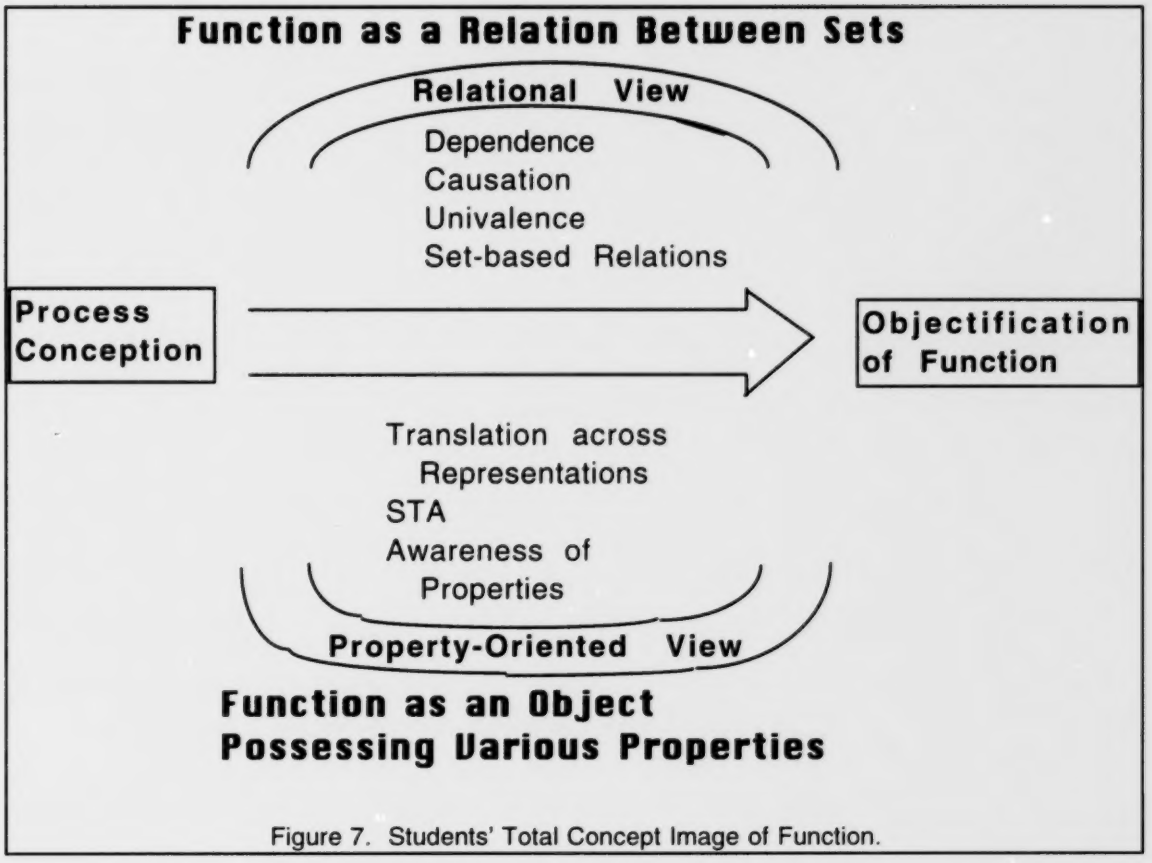


Figure 7. Students' Total Concept Image of Function.

their greatest increase on tasks which implicitly called for graphical analysis. An analysis of the problem solving strategies on the written tests provided evidence that students' uses of the GC allowed them to approach tasks with a wealth of available procedures. The ability to relate functions and functional properties in different representations provided evidence that a View 2 structural conception was forming. Hence, evidence was found to indicate the following effects of the GC on student learning:

1) The GC facilitated multi-representational problem solving strategies.
2) The graphic images of the GC were a factor in strengthening the students' structural views of function.
3) The students increased their use of the GC the most over the course of the year on tasks which implicitly called for graphical analysis.

However, the following misconception was also found in the case study participants, and was probably due to the use of the GC:

1) The students' concept image of the graph of a function involved continuous functions with infinite domains, and which could be represented by a symbolic expression of the form $y = f(x)$.

Very few examples of non-elementary functions were used as examples throughout the course. Hence, the students almost exclusively thought about the properties of polynomial and exponential functions since they were the only ones to which they were exposed. This presents educators with a dilemma in regard to the use of the graphing calculator, as well as some other types of graphing software. On one hand, we would like students to be able to generate their own views of function, their own examples of function, and to discuss the variety of functions which occur in everyday life. Many of these examples, however, are not readily produced using the GC, and some do not even have approximations which would readily be produced. The price of a postage stamp, the velocity of a plane during a flight, the water level of a draining sink, and the temperature inside a refrigerator are all examples of functions which would be quite difficult to produce using a GC. Graphing software, especially the GC, can be very restraining on the amount of student investigation it allows with regard to a variety of function exemplars.

Therefore, even though data previously presented showed that the GC can be a boon to the students' property-oriented development (both on a local and global basis), it also may restrict the students' View 2 conception in regard to the range of properties a function is allowed to take on. The case study participants refusal to accept as functions those equations which could not be "solved for y", as well as those graphs which were discontinuous or had unusually restricted domains, provides support for this claim.

Overall, it appears that the GC positively influenced the learning outcomes of the classroom in this study. Additional long-term studies which investigate the effects of graphing software (including the GC) on instruction and learning are greatly needed. These studies should focus on all aspects of the students' concept image, including computational/relational distinctions as well as the formation of procedural networks across function classes and representations.

Table 1
Procedural and Relational Distinctions in Students' Definitions of Function throughout the Year

	Pretest 1 (n = 16)	Posttest 1 (n = 17)	Pretest 2 (n = 33)	Posttest 2 (n = 30)	Pretest 3 (n = 33)	Posttest 3 (n = 34)
Procedural References	12	15	8	10	11	6
Relational References	3	12	23	17	20	28
Neither	1	0	2	3	2	0

Table 2
Procedural and Relational Distinctions in Students' Discussions of Function throughout the Year

	Pretest 1 (n = 16)	Posttest 1 (n = 17)	Pretest 2 (n = 35)	Posttest 2 (n = 28)	Pretest 3 (n = 36)	Posttest 3 (n = 34)
Procedural References	12	10	20	18	19	20
Relational References	3	7	9	7	13	12
Neither	1	0	6	3	4	2

Table 3
Function Questionnaire Item on Procedural (computation) versus Relational (correspondence) Aspects of Function

	October (n = 16)	May (n = 34)
A function is a computational process which when given one value of a variable (normally x) produces one value of another variable (normally y).	94% (15)	85% (29)
A function is a correspondence between values of one set and values of another set.	6% (1)	15% (5)

Table 4
GC Questionnaire Items on Students' Use of the Graphing Calculator

	October (n = 17)	May (n = 34)
I prefer to use the graphing features of the calculator for problems which ask me to graph a function.	87% (14)	91% (31)
I prefer to graph by hand problems which ask me to graph a function.	13% (2)	9% (3)
I prefer to use the graphing features of the calculator for problems which ask me to analyze a function.	76% (13)	71% (24)
I prefer to solve by hand problems which ask me to analyze a function.	24% (4)	29% (10)
I prefer to use the graphing features of the calculator for most problems which do not explicitly ask me to create a graph or analyze a graph, but where a graph may help.	59% (10)	85% (29)
I prefer to solve by hand most problems which do not explicitly ask me to create a graph or analyze a graph, but where a graph may help.	41% (7)	15% (5)
I use the graphing features of the calculator when working many of my homework problems.	82% (14)	82% (28)
I use the graphing features of the calculator when working very few of my homework problems.	18% (3)	18% (6)
I solve many problems with the graphing calculator which I could not solve otherwise.	47% (8)	71% (24)
I solve very few problems with the graphing calculator which I could not solve otherwise.	53% (9)	29% (10)

Table 5
Percentage of Student Use of the Graphing Calculator on Written Test items of UI#2

Yes = Yes, I used the graphing features of the graphing calculator in the solution of this problem.

No = No, I did not use the graphing features of the graphing calculator in the solution of this problem.

	Pretest (n = 35)		Posttest (n = 36)	
	Yes	No	Yes	No
Graph $f(x) = x^5 - 81x$	97%	3%	100%	0%
Graph $f(x) = -x^5 + 12x^4 + 10x^3 - 120x^2 - 9x + 108$	87%	13%	100%	0%
Find the zeroes of $f(x)=x^3-7x-6$	79%	21%	97%	3%
Find the zeroes of $f(x) = x^2 - 8x - 20$	37%	63%	63%	37%
Find all values of x such that $5x^2 + 9x - 3 > 4x + 5x^2$	14%	86%	25%	75%
Factor $f(x) = x^3 + 3x^2 - 4x - 12$	32%	68%	41%	59%

Table 6
Percentage of Student Correctness on Written Test items of UI#2

	Pretest (n = 35)		Posttest (n = 36)	
	Correct	Incorrect	Correct	Incorrect
Find the zeroes of $f(x)=x^3-7x-6$	83%	17%	88%	12%
Find the zeroes of $f(x) = x^2 - 8x - 20$	89%	11%	94%	16%
Find all values of x such that $5x^2 + 9x - 3 > 4x + 5x^2$	77%	23%	78%	22%
Factor $f(x) = x^3 + 3x^2 - 4x - 12$	48%	52%	56%	44%

Table 7
Student Responses on Graph Construction Items on Written Tests of UI#2

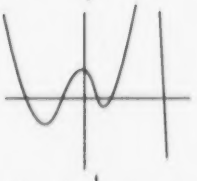
$f(x) = -x^5 + 12x^4 + 10x^3 - 120x^2 - 9x + 108$	Pretest (n = 35)	Posttest (n = 31)
	4	7
	19	20
	3	3
	2	1
	3	0
Blank	4	0

Table 7 (cont.)
Student Responses on Graph Construction Items on Written Tests of UI#2.

$f(x) = x^5 - 81x$	Pretest (n = 35)	Posttest (n = 31)
	20	28
	5	3
	5	0
Other	4	0
Blank	1	0

Table 8
Percentage of Student Use of the Graphing Calculator on Written Test items of UI#3

Yes = Yes, I used the graphing features of the graphing calculator in the solution of this problem.

No = No, I did not use the graphing features of the graphing calculator in the solution of this problem.

	Pretest (n = 35)		Posttest (n = 36)	
	Yes	No	Yes	No
Find the x-intercepts of $f(x) = 2^x - 4$.	76%	24%	71%	29%
Graph $f(x) = 2 \cdot 10^x$	97%	3%	91%	9%
The graph to the right represents the population P(x) of a U. S. city, where x is the number of years after 1990 (for example, x=3 corresponds to the year 1993). According to this graph, the city's population was 15,000 in 1990. In what year do you expect the population to reach 30,000?	6%	94%	9%	91%
$5000 is invested in a bank at 5% interest compounded continuously. The amount of money A(t) after t years is given by the formula $A(t) = 5000e^{.05t}$. How many years will it take for the investment to double?	42%	58%	44%	56%
Harry's Hamburger Stand had 40 customers during its first week of operation. The deliciousness of Harry's hamburgers spread across town, and the number of customers during the first five weeks of operation are given in the table to the right. Using this information, how many customers do you expect Harry to have during the sixth week? Explain your reasoning.	56%	44%	44%	56%

Table 9
Identified Translation Strategies used by Keith During Each Unit Investigation

Interview	Strategy	ET	GT	CE
Pretask #1				
	Local			
	Global			
	Combination			3
Posttask #1				
	Local	6	2	
	Global	1	1	4
	Combination			
Pretask #2				
	Local	5		
	Global			5
	Combination			
Posttask #2				
	Local	3		1
	Global			4
	Combination			
Pretask #3				
	Local	2	2	
	Global		1	3
	Combination		1	
Posttask #3				
	Local	5	1	
	Global			1
	Combination	1	1	1

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